RELATIVITY AND COSMOLOGY I

Problem Set 12 Fall 2022

1. The Reissner-Nordström Black Hole

The Reissner-Nordström metric is the static spherically symmetric solution to Einstein's Equations sourced by the stress tensor of electromagnetism

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,, \tag{1}$$

where Q is the electric charge of the Black Hole and we are working in natural units $G = 1$. For stationary metrics, the horizon is found where[1] $g^{rr} = 0$.

Case 1: $M^2 < Q^2$

(a) Show that, if $M^2 < Q^2$ this spacetime has a naked singularity [2].

(b) Argue that no physical collapse can produce a Black Hole with $M^2 < Q^2$.

(c) Without doing any calculation, can you guess what the Penrose diagram of this spacetime looks like?[3]

Case 2: $M^2 = Q^2$

This represents a so-called **extremal Black Hole**.

(d) Change coordinates to $\varepsilon = r - |Q|$. Write down the metric and expand around $\varepsilon = 0$. Argue that this is the near-horizon limit.

(e) Make one more change of variables to $z = \frac{Q^2}{\varepsilon}$. You should obtain the following metric

$$ds^2 = Q^2 \left[\frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}\Omega^2\right] . \tag{2}$$

You should recognize that the first term in the square brackets is the Lorentzian equivalent of the Poincaré half plane you saw in Problem Set 7. Those coordinates cover part of what is called **Anti-de Sitter** (AdS) space.

Case 3: $M^2 > Q^2$

(f) Show that this spacetime has an inner r_- and an outer r_+ horizon.

(g) Compute the area A of the outer horizon[4] Write explicitly its variation $dA(M, Q)$.

[1] See discussion on Carroll section 6.2.

[2] Use the fact that $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is singular as $r \to 0$ without proving it.

[3] **Hint**: Consider how the metric looks as $r \to \infty$ and the fact that the singularity is naked.

[4] **Hint**: use the expression you found for r_+ and use spherical symmetry.

(h) Invert it to find $dM(A,Q)$. Match it to the following form

$$dM = \frac{\kappa}{8\pi} dA + \Phi_H dQ \,, \tag{3}$$

where κ is called the **surface gravity**. Can you give an interpretation for Φ_H?

(i) Bekenstein and Hawking argued that Black Holes should have a thermodynamic entropy S related to the area of the horizon A (in natural units) as

$$S = \frac{A}{4} \,. \tag{4}$$

By staring at (3), can you identify the temperature of the Reissner-Nordström Black Hole?

(j) Compute the asymptotic behavior of $T(Q,M)$ as $M \gg Q$.

2. Penrose Diagram of de Sitter and Anti-de Sitter

The three Lorentzian maximally symmetric vacuum solutions to Einstein's equations are Minkowski, de Sitter (dS) and AdS. They are respectively solutions when the cosmological constant is $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$. Last week we found the Penrose diagram of Minkowski space. This week, we study dS and AdS.

dS This spacetime is of relevance to cosmology, because it describes the metric of our universe on the largest scales when the energy density of the universe is dominated by dark energy. In Problem Set 6, you saw that cosmological spacetimes can be described by the following class of metrics

$$ds^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \,, \tag{5}$$

where in a period of dark energy domination $a(t) = e^{Ht}$, corresponding to dS. As for the Schwarzschild metric, these coordinates can actually be extended until they cover the full manifold. After doing that, we obtain dS in global coordinates

$$ds^2 = -\mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}\Omega_3^2 \,, \tag{6}$$

where $\mathrm{d}\Omega_3^2$ is the metric on a 3-sphere and $\tau \in (-\infty, \infty)$.

(a) What is the explicit form of $\mathrm{d}\Omega_3^2$?

Now let us compactify time to find the Penrose diagram. We perform the following change of coordinates

$$\cosh\tau = \frac{1}{\cos T} \,. \tag{7}$$

(b) Find the metric in the new coordinates and identify the range of values taken by T. What is the spacetime that is conformally related to dS according to this metric you found?

(c) Draw the Penrose diagram of this spacetime by putting T on the vertical axis and one of the angular coordinates on the horizontal axis. What is the shape of the diagram? Why is it different than Minkowski?

(d) From the Penrose diagram of Minkowski, you saw that an observer at future timelike infinity has the full spacetime in its past lightcone. What can we say about an observer in dS?

(e) Is there an event horizon in this spacetime?

AdS The metric in global coordinates reads

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_2^2 \,, \tag{8}$$

with $\rho \in (0, \infty)$ and $t \in (-\infty, \infty)$.

(f) Perform the change in coordinates $\cosh\rho = \frac{1}{\cos\chi}$. What is the spacetime that is conformally related to AdS? What are the ranges of its coordinates?

(g) Draw the Penrose diagram. What is the structure of spatial infinity?

(h) Write the energy conservation law in AdS for timelike radial geodesics in the coordinates in (8). What can we say about particle motion in AdS? Can a particle ever reach $\chi = \frac{\pi}{2}$? What about photons?